Invited Lecture

Arguments or Findings Regarding Language As Resource For Mathematics Learning And Teaching

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ABSTRACT Language as resource is a challenging research approach in mathematics education because it examines how language can function to support mathematics learning and teaching. The approach originally started to develop in response to discourses of non-mainstream languages and cultures as problems or obstacles to mathematics teaching and learning. In this text, I revisit and bring together four empirical studies in order to discuss four major findings that are arguments to explain the complexity and importance of the language as resource approach. These four arguments are: 1) the huge potential of all languages to make mathematical meaning; 2) the critical realization of some languages in the mathematics classroom; 3) the critical communication of mathematical meaning in classroom teaching talk; and 4) the huge potential of teaching talk to support mathematics learning for understanding.

Keywords: Language as resource; Sociocultural language-based stances; Mathematics learners and learning; Mathematics teachers and teaching.

1. Language for/as Communication and Language as Resource

I remember very well my years as a secondary school mathematics teacher. In 1997, I met Sergev, the fifteen-year-old son and brother of a Bosnian migrant family who had arrived in Barcelona less than one year earlier. He spoke a little English and Spanish, and three days per week attended the special lessons in the school for students who were in the process of learning Catalan, the language of instruction. For the rest of the days, I was his mathematics teacher in the ordinary classroom where he seemed to enjoy doing mathematics. One day, another student solved a quadratic equation on the board and wrote down the numerical solution '3', without any sign. She had explained in Catalan that the negative solution made no sense in the context of the word problem given. When the bell rang, Sergev remained in his seat and I approached him. He then went to the board, pointed at the number symbol 3, and used Catalan, Spanish and English, none of which were his home languages, to share the following observation:

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> *A l'escola meva este número es siempre positive. Aquí es positive i negative.* [In my school this number is always positive. Here this number is positive and negative.]

Sergev was trying to adapt what he knew about mathematical symbols and quadratic equations to figure out why there was only one number written as a solution. I thought that he had possibly not understood or made sense of the explanation in Catalan regarding why only the positive solution had been written on the board. Since he knew there should be two solutions, he reinterpreted the number 3, with no sign accompanying it, as a way of representing both the positive and negative solutions of the equation. We engaged in Catalan-Spanish-English mathematical talk about the negative solution and in reasoned conversation regarding what counts as a solution of an equation that is modeling an everyday context. Over the course of the conversation, Sergev insisted on the distinction between the solutions of the equation and the solution of the problem. Multilingual talk, rather than linguistically accurate talk in just one language, created opportunities of mathematics teaching and learning other than those possibly created by the written symbols and algebraic expressions. Together, we managed to make talk work as a resource at the service of communication and mathematical understanding. Our flexible use of language allowed us to move from how to proceed to solve a quadratic equation towards what counts as a solution.

The episode with Sergev illustrates in practice the basic idea that mathematics teaching and learning involves the successful use of *language for/as communication* (for the communicative approach to mathematics teaching and learning, see, e.g. Morgan et al., 2021; Pimm 1987/2021; Planas et al., 2018). We developed spoken communication that helped us to overcome our respective lack of linguistic knowledge of the first languages of each other, and to engage in reasoned mathematical discussion. Sergev and I experienced both language *for* mathematical communication and language *as* communication of mathematical understanding, even if at times this could be limited communication. In close connection to these notions of language for/as communication, *language as resource* (Planas, 2018, 2019) plays an important role in classroom research on mathematics and language, with the meaning for 'resource' assumed to establish a discourse of language as supportive and respectful of school learners, their learning needs, and the values of their cultures, communities and funds of knowledge. There is, however, a diversity of interpretations regarding what is meant by this notion or approach and what it adds to the communicative approach, influenced by the diversity of interpretations regarding what language is. Many variables affect the choice of how to define 'language'.

1.1. A sociocultural understanding of language

In the context of multilingual classrooms, there is a tendency to start at the level of entire 'national' language systems, or the set of linguistic resources that children first develop in their homes. This is a problematic perspective that leads to a paradox or tension that is difficult to solve. Decisions about what counts as a distinct language are political: do we count varieties of South American Spanish, of Castilian, of Indigenous Creole Spanish, of…? In order to further define language and to work out aspects of specific varieties of language, research in mathematics and language tends to focus on special cases of linguistic peculiarities, such as specific groups of people who supposedly live under similar linguistic-cultural conditions, for example people who migrated from South American regions to Catalonia. This form of analysis keeps producing normative attributions of equality within groups and categories of difference are created, despite the attempt to overcome them. This paradox does not seem to be solvable and can only be countered by permanent awareness of the inevitably reductionist access to any group of people according to linguistic-cultural backgrounds. The problem of distinguishing between varieties of languages extends to distinctions between 'everyday' registers or languages (whose 'everyday'?) and mathematical languages (whose 'mathematics'?). Once the scope of the definition of language is determined, often with reasons for limiting the extent of multimodality, the question still arises as to what units of language are of interest. Linguists conventionally distinguish nested ranks of grammatical units such as morpheme, word, phrase, clause and sentence, in combination with levels of analysis such as utterance, interaction, text, genre, register and discourse. Depending on which units are privileged, mathematics education research on language addresses the processes of mathematics teaching and learning very differently.

In my work, I adopt sociocultural stances in the understanding of language not just as a tool for communication, but as one of the most important means of experiencing, interpreting and shaping the worlds around us through multiple processes of meaning making (Gee, 2004; Halliday, 1985; Wells, 2009). Language is thus part of what constitutes mathematical learning, and is made visible as learners talk among themselves or with others, and in how they communicate meanings and make sense of them. In line with this definition of language, Planas (2018) conceptualized the language as resource approach in classroom mathematics research by building on two arguments: 1) the huge potential of all languages to make mathematical meaning; and 2) the critical realization of some languages in the mathematics classroom. In that paper, I also claimed that enhanced talk amongst diverse languages and meanings in the construction of the language of the classroom would yield further opportunities to learn school mathematics. Since then, broadening the research to include the focus on the languages of teachers in mathematics content teaching has provided substantial insights by bringing up two more arguments: 3) the critical communication of mathematical meaning in classroom teaching talk; and 4) the huge potential of teaching talk to support mathematics learning with understanding. In the rest of this paper, I elaborate on these four arguments by presenting data and findings from four classroom studies conducted in schools of Barcelona over the last two decades. Collaborations with colleagues in other countries and continents with similar research interests suggest that the findings seen in this particular context are also distinctive of mathematics teaching and learning in a larger quantity of classrooms, this being typical of the arguments that support the importance of the language as resource approach.

The two majority languages spoken in Barcelona are of the Romance family: Catalan and Castilian, as Spanish is known in this part of Europe. Like in many other parts of the world, the language policy in education has long been an issue, with a monolingual orientation that values one language over others. The Catalan-only language policy is used as a means to address social inclusion since it is expected to help integration and ease historical tensions coming from various decades of Catalan being forbidden (for recent discussion of the political and linguistic intricacies of the case of Catalonia, see, e.g. Aramburu, 2020). As would occur with a monolingual Castilian-only policy, several challenges are posed to school students who are not expected to learn mathematics through their home languages, as well as to teachers who are not expected to promote languages other than Catalan in classroom interaction. A consequence is that while the language policy in education was created to address social inclusion, it has unintended effects on different language groups by increasing the gap in their access to classroom participation and mathematics learning opportunities. This is the context in which I conduct research and developmental work, in the belief that all students, regardless of their languages, cultures and socioeconomic backgrounds, have the right to learn mathematics with understanding.

2. The Huge Potential of All Languages to Make Mathematical Meaning

I start with the argument first reported in the field literature, regarding the recognition that all languages have the potential to support mathematics teaching, learning, and content meaning making. Mathematics education research guided by this argument and built on views of language as social, challenges monolingualism as the acceptable norm (Planas, 2021a), and invites research and practice that consider the students' home languages and how they are present in and mediate the generation of mathematical meaning across discourse practices. Two decades of findings in classroom research on mathematics and language (Planas et al., 2021; Essien and Msimanga, 2021) reveal that multilingual students engage in multilingual practices that enable them to complete mathematical tasks given or presented in the language of instruction. The successful use of all the languages at their disposal, through flexible language switching and translanguaging (Sapire and Essien, 2021) has been proved to be common in situations of working on meaning in order to clear up misunderstandings, follow up more complicated mathematical reasoning, and ask one another questions.

Drawing on the argument that all languages are potential resources for mathematics learning, Planas and Setati (2009) reported strategies of bilingual migrant students in Barcelona in order to successfully deal with mathematical tasks introduced in the language of instruction. With minimal pedagogical intervention from the classroom teacher, bilingual students born in South America switched languages to provide continuity to small group work by means of selecting one language and then shifting to the other depending on the conceptual complexity of the task. In that study, the collection and analysis of lesson data was aimed at investigating shifts between Catalan [C] and Spanish [S] in mathematical communication. The research questions were: Do Spanish-dominant bilingual students in Catalan classrooms switch languages during mathematical activity? If so, what are some of the factors that seem to promote language switching with a group of these students in specific lessons? An assumption framing these questions was that the contrast between mathematical participation in a small and the whole group was related to the language mostly spoken in the interaction.

All the lesson transcripts were analyzed using a version of sociocultural discourse analysis (Gee, 2004; Halliday, 1978, 1985), that is, an analysis being concerned with the content, form and function of spoken language and the processes through which shared meaning and understanding are developed in social context. This included coding themes and quantifying and comparing code frequencies to arrive at patterns which were then exemplified by lesson extracts that showed language forms and functions across participation in mathematical activity. The focus of the analysis was therefore on both the activity around mathematical content and the language forms in use. The flexible switching between Catalan and Spanish in student interaction, and hence the permeability of mathematical participation through these two languages, was one evident pattern. The analysis allowed us to uncover a second pattern which served to corroborate our initial assumption. We detailed the systematic change to the home language when there was also a change in the nature of discourse from descriptions of observations and procedures to mathematical explanations. We thus concluded that the students switched to their shared home languages as soon as the conceptual demands in mathematical talk increased. Finally, these were the two patterns specified:

- Bilingual students flexibly use their languages when engaged in describing observations and procedures in small group work.
- These students tend to use their common home languages when engaged in conceptual building and mathematical explanations.

As an insight into bilingual mathematics learning and creative translanguaging practices in small group work, Table 1 presents a transcript of data from Planas and Setati (2009). It belongs to the part of a secondary school lesson unit called "Our dynamic planet," which included mathematical activities that encouraged students to pose questions and solve problems in real contexts. This unit had been designed the year before by a group of teachers in the school as part of the development of teaching materials in support of mathematics learning with understanding. In the third lesson, the teacher wanted the students to think about "How can you mathematically represent a tornado?" From the transcript below, it is clear that language switching between Catalan and Spanish occurs for communication and discussion on the possibilities of graphically representing, through combinations of plane isometries, the spatial movements that make up a tornado. Máximo and Eliseo use their home language to discuss the arrow representing the perpendicular movement around the tornado axis and its orthogonal axis (horizontal and vertical) on the task sheet. The group discussion, of which the transcript in Tab. 1 is a fragment, follows a sequence of descriptions of characteristics of a tornado and explanations of mathematical representations through reasoning and concept building around composition of isometries.

Tab. 1. Transcript of lesson interaction (Planas and Setati, 2009, pp. 43, 48)

Máximo:	[C] *Hem de decidir les fletxes que dibuixem i ja està.* [We need to decide the arrows that we draw and that's all.]
Eliseo:	[C] *Primer pensem les fletxes, després les dibuixem i després en parlem.* [First we think about the arrows, then we draw them and then we talk about it.]
Máximo:	[S] *Esta idea de las flechas no es fácil. Tenemos que imaginar los diferentes movimientos que existen dentro del tornado.* [This idea of the arrows is not easy. We have to imagine the different movements that exist within the tornado.]
Eliseo:	[S] *Una flecha tiene que ser una línea recta para que el tornado baje. Tenemos la t para la translación.* (…) [An arrow needs to be a straight line for the tornado to go down. We have the t for the translation.] (…)
	[C] *El que hem de fer és entendre què és un tornado i després li busquem un nom.* [What we need to do is to understand what a tornado is and then we find a name for it.]
Luna:	[C] *Hem de fer les fletxes com ahir?* [Do we need to make the arrows like yesterday?]
Eliseo:	[C] *El que hem de fer és entendre què és un tornado i després li busquem un nom*. [What we need to do is to understand what a tornado is and then we find a name for it.]

The nature of the mathematical discourses and how bilingual or multilingual students engage in them through bilingual practices has been researched in various other contexts in which English is the language of instruction. González, Andrade, Civil and Moll (2009) with Latino bilinguals in Arizona, and Setati and Adler (2000) with multilingual students in South African townships, characterized language switching as a successful strategy for convergence towards school mathematics in the English language of instruction. Language switching was claimed to be a 'natural' unproblematic effect of what multilinguals do with language in mathematics classrooms, even when the educational policies and classroom norms are differently oriented and refrain students from flexibly using their languages to communicate their thinking. It was additionally claimed that multilinguals within mathematics classrooms behave as people who speak more than one language generally do, and that language switching does not necessarily stand for lexical gaps, linguistic difficulties or deficient language abilities. Like in Planas and Setati (2009), these studies show classroom discourse to be the site in which everyday languages, the school language in general and the school mathematics language in particular become connected.

3. The Critical Realization of Some Languages in the Mathematics Classroom

The episode with Sergev in the introduction suggests the role of the teacher in establishing a favorable climate for using all the languages available at the service of communication and mathematics teaching and learning. Establishing such a favorable climate is, however, not straightforward. In line with this, the second argument refers to the different valorization of the languages in the mathematics classroom, and hence the different distribution of mathematics learning opportunities amongst their speakers. In my context, for example, influential ideologies underlying language policy situate Castilian as a language that is more valuable than other forms of Spanish spoken by South American migrants, who tend to be seen as powerless groups of people (Planas, 2021a). Far from viewing language as a neutral object in the classroom, it is therefore necessary to address questions concerning the several visible and invisible messages that are sent to students (who are in particular language users) through the differing valorizations of languages (and language uses). It is complex to know whether language uses provoke valorizations or valorizations provoke language uses. Both directions are at the heart of debates concerning multilingual mathematics classrooms: Do students facilitate particular positions in the classroom by the fact of using a language in their discussions with others at certain moments? Is it that talking with some of the other participants in the classroom leads to the use of a certain language alongside the creation of particular positions? The debate about what comes first, however, is not directly related to the educational and pedagogic debate.

As said by one of the teachers in Barcelona, the situation is more complex than just letting South American migrant students use their home languages in the classroom. In Planas and Setati-Phakeng (2014), valorizations between the languages and their users, on the one hand, and the mathematical discussions, on the other, were documented in a variety of secondary and primary school lessons and in interviews with mathematics teachers in the urban areas of Barcelona, Johannesburg and Pretoria. Like in Planas and Setati (2009), in that study, the data were transcribed and analyzed using a version of sociocultural discourse analysis to examine how teachers used talk (in the classroom lessons and the interviews) to represent their students as learners and knowers of mathematics. A combination of qualitative and quantitative methods enabled the study of how the realization of the languages of students to do mathematics was mediated by specific valorizations of the teachers. Again, this implied coding themes and quantifying and comparing code frequencies to find commonalities which were then exemplified by lesson and interview extracts that showed language forms and functions across or about participation in mathematical activity. On this occasion, the analysis allowed us to uncover the following major themes or findings:

- [From the interviews] Teachers referred to students whose home languages and knowledge were not helping them to speak mathematics accurately.
- [From the lessons] Teachers focused on the mathematical content but also on the linguistic accuracy in the talk of the students in the interaction.

The transcript below illustrates a classroom conversation between the teacher and Luis, a migrant student from South America whose knowledge of the language of instruction was good (Tab. 2). Luis is provided the opportunity to learn both that *senar* is the Catalan name for the notion of odd number, and that his statement of a mathematical property is relevant to the understanding of the notion. During group work aimed at talking $2x + 1$ with words, the use of specialized vocabulary was required. Odd [number] is not a priori a difficult word but the Catalan name for it, *senar*, is quite different from the Spanish name, *impar*. Instead of elaborating on the arithmetic meaning that can be given to an algebraic expression, the teacher interrupted the mathematical discussion started by Luis, who had found a way of referring to the category of odd numbers, ‘‘even plus one,’’ to put the focus on the language of instruction. This teacher acted differently in other turns: ‘‘So you don’t spend time on these things”, or ‘‘Even plus one is a good property to remember.’’ The focus on the names in the language of instruction somehow unfocused Luis’ thinking of odd numbers versus even numbers, both as relative to the grouping process with 2 as the number to group by, and 1 or 0 as the remainders.

Tab. 2. Transcript of lesson interaction (Planas and Setati-Phakeng, 2014, pp. 888−889)

Luis:	[C] *He estat molt temps perquè no podia recordar la paraula en català de* [S] *l’impar.* [I spent a lot of time because I couldn’t remember the Catalan word for odd.]
Teacher:	[C] *Doncs no perdis temps en aquestes coses, busca una altra manera de dir-ho.* [So you don’t spend time on these things, you find another way to say it.]
Luis:	[C] *Umm… Puc dir* [S] *par más uno?* [Umm… Shall I say even plus one?]
Teacher:	[C] *Ara que saps la paraula, eh?… senar, digues-la.* [Now that you know the word, eh?… odd, you say it.]
Luis:	[C] *I ara també sé una altra manera, per si em torno a oblidar de la paraula.* [And now I also know a different way, just in case I forget the word again.]
Teacher:	[C] *Millor que no la oblidis. És una paraula important en matemàtiques. I sí! parell més u és una bona propietat per recordar, ser consecutiu d’un nombre parell.* [You’d better not forget it. It’s an important word in mathematics. And yeah, even plus one is a good property to remember, that of being consecutive to an even number.]
Luis:	[C] *I millor que no oblidi la propietat!* [And I’d better not forget the property!]
Teacher:	[C] *I tant! És més important arribar a conèixer la propietat.* [Yeah! It’s more important to get to know the property.]

Although the home language of Luis was heard in the small group discussions and in social talk in his school, it was not the language of whole class communication. Students like Luis were able to participate in many aspects of the mathematics classroom but for him to participate in the whole class discussions he had to be singled out by the teacher explicitly switching to Spanish or inviting him to participate by accurately using the language of instruction. In our observations we did not witness any instances of Spanish dominant students in classrooms where they were in a minority speaking up in Spanish in front of the whole class, unless the teacher had directed the question to them. Although this singling out was probably well received in the schools in Barcelona due to the affiliation to Spanish and the overall bilingual environment, still we wonder about how students in this age group perceive being treated differently from their peers through a language switch or through mentions to their learning of the language of instruction, e.g., “Now that you know the word…”

There is no doubt that language ideologies have an impact on the students’ lives at multilingual schools and on their mathematical learning. Learning is often judged from the way it is communicated, and communication has a great deal to do with languages. Such ideologies are instilled so deep inside a society that students sometimes anticipate what will be the effects of certain uses of their languages, and thus rearrange their communication opportunities. Research on this topic has stated the sociopolitical dimension of learning mathematics in multilingual classrooms, but also in classrooms with students who are linguistically disadvantaged for a variety of reasons, including impoverished socioeconomic status. In the example with Luis both constraints and opportunities can be uncovered in how this student goes on with his mathematical learning in a classroom in which the language of instruction is not a home language.

4. The Critical Communication of Mathematical Meaning in Classroom Teaching Talk

The third argument or set of findings around the complexity and importance of the language as resource approach refers to the fact that the communication of mathematical meaning in classroom teaching talk is not always sufficiently explicit or precise. Communication in the mathematics classroom, as in all other language contexts, is made up of communicative intent and intended meaning on the one hand, and communicative effective function and interpreted meaning on the other. Mathematics teaching and teachers need to support students in their communication of the intended mathematical meaning, but also need themselves to successfully resolve their communicative intents of mathematical meaning. One of the challenges with mathematics teaching talk is that the intended content meaning is not always communicated as clearly as expected.

In Planas (2019, 2021b), this discussion was addressed for the case of algebraic contents through the analysis of a number of instances of two teachers’ talk in their school lessons. The analysis was guided by the identification and interpretation of classroom teaching talk with meaning potential to support the communication and learning of specific algebraic contents. In the teaching of equations, for example, teaching talk with potential to support the communication of specialized meaning for the concepts of algebraic equivalence and equal sign was revealed as both crucial and critical. Once more, a version of a sociocultural discourse analysis particularly framed within Functional Linguistics (Halliday, 1978, 1985), allowed the construction of two main themes:

- Teachers give names and explanations with potential to support the communication of meaning for specific mathematical contents.
- The use of some other names and explanations in teaching talk could have increased the opportunities of communicating important content.

These findings were corroborated in a recent investigation (Planas, Alfonso & Rave-Agudelo, submitted) with data from seven transcripts of lessons of four secondary school teachers and two transcripts of one primary school teacher. These were all representations of content lessons from design-based studies in which I had collaborated with teachers and doctoral students in seminars to create the tasks. These data were now imported to identify quality aspects of teaching talk at the word and sentence levels of language. The first column of Table 3 lists the five mathematical contents that were the object of learning in the one or two lessons with each of the five teachers. In the classroom studies, the teaching had not been prepared or planned to meet specific learning challenges. To assess teaching talk, we took decisions as to which content learning challenges might need support in talk oriented to teach the curricular content in play. We selected two learning challenges per content from the specialized literature; for this, we were further advised by five colleagues with expertise in those areas of research and who were knowers of the local mathematics curricula and school system. We realized that names and explanations in teaching talk can be interpreted by considering how they support the overcoming of content learning challenges which are highly predictable regardless of whether or not manifested in school students' talk. Since the lesson transcripts included students' talk, we explored both cases. This distinction is not trivial given the need in teaching talk to support content learning challenges even when their manifestations cannot be traced in the available products of the students. Some or even many of the learners might be experiencing important content learning challenges selected from the specialized literature that were not made public in talk or written products.

Tab. 3. Elements and examples of teaching talk (Planas, Alfonso, and Rave-Agudelo, submitted)

Content	Challenge	Names and Explanations
Probability	Equiprobability bias (Green, 1982)	It is possible or impossible, not the most possible, but it can be *the most likely.* // These eleven outcomes are *not equally likely or equally probable.* // *From impossible to certain,* we can find *very unlikely, not likely, likely…*
	Representativeness bias (Tversky & Kahneman, 1973)	This sequence may be *random-looking* but it is *not more probable.* // You can follow a random procedure and obtain an *ordered-looking* result. // Do not judge *more frequent* and *more probable* as if they were more familiar.
Areas	Area-perimeter confusion (Stavy & Tirosh, 2000)	Two-dimensional figures have *enclosing lines or perimeters* and also have *areas or enclosed regions.* // Let us think of *the smallest area for this perimeter.* // *Numbers related to areas* can be smaller than *numbers related to perimeters*.
Decimals	Whole number bias (Resnick et al., 1989)	By adding more *decimal places*, we do not make a number larger. // 0.61 is smaller than for example 0.62 because 0.61 is *one hundredth less*. // The smallest number has the greatest *number of digits after the decimal point.*
Algebraic equation	Equal sign misconceptions (Kieran, 1981)	The *equal sign* in the equation is a *relational symbol for equivalence*. // *Equal* indicates *balance* between the expressions on each side. // On both sides of the *equal sign* you write *equivalent expressions.*
Triangles	Prototypical thinking (Clements & Battista, 1992)	*A triangle height* is *an either internal or external segment.* // The three *height feet* are not any point of each of the three sides. // It is not any angle either; it is *the perpendicular angle.*

The second column of Tab. 3 is for the content learning challenges whose consideration was traced in the transcribed talk of the teacher. For example, names and explanations with meaning potential for reducing equiprobability biased thinking were traced in teaching talk throughout the secondary school lesson where the experiment of throwing two dice and summing up the numbers was presented. In the lesson transcript, we found an explanation by the teacher in response to some evidence of biased thinking:

> Student: It is like throwing one die, but now you have more to choose, from two to twelve, and so the probability to guess is one over eleven.
> Teacher: These eleven outcomes are not equally likely or equally probable.

Prior to this conversation, the teacher had given explanations with important names whose meaning potential supported debiased probability reasoning, such as: "It is possible or impossible, not the most possible, but it can be the most likely." For the representativeness bias, we also traced some support in the form of content names and explanations in the teacher talk over the two lessons. The other four contents and five lessons each had two associated learning challenges and, for all of these lessons, clear support in the form of content names or explanations in teacher talk could not always be traced. The third column of Table 3 reproduces examples of content names (in bold) and explanations of importance for supporting all learners, especially those who face the associated learning challenge.

5. The Huge Potential of Teaching Talk to Support Mathematics Learning for Understanding

The fourth argument or set of findings regarding the complexity and importance of the language as resource approach refers to the enormous potential of teaching talk to support mathematics learning for understanding (or mathematics learning that implies leveraging multiple mathematical meanings and connections amongst them). Given the critical communication of specific mathematical meaning in classroom teaching talk, however, the realization of this potential cannot be taken for granted. Teachers need to develop professional expertise about how students use language to learn mathematics, and about how to use language to teach mathematics. For this, they need time and space to practice and work collaboratively with others towards productive mathematical talk in teaching. In order to improve the impact of mathematics professional development in classroom practice, increasing attention has been paid to work with teachers guided by teaching needs (Kazima et al., 2016), which may vary across cultures and groups of teachers (Essien, Chitera and Planas, 2016).

In this section I present the study conducted with two secondary school teachers, Jana and Maia, in the first round of the research and developmental project introduced in Planas (2019, 2021b). These teachers had expressed various concerns with the teaching of equations, which was an icon of mathematical knowledge in their schools with some of the families especially interested in test results on this content. They had several years of mathematics teaching experience, and worked in two different schools of Barcelona at the time of the study. The results of their students in the annual tests for the past years had shown poor conceptual understanding of equations, in contrast to the good performance in the resolution of equations and in the translation of word problems into algebraic expressions. Poor conceptual understanding was revealed in the beliefs that: two different equations can have the same numerical solutions; and an equation can be simplified into numerical solutions without an operation sign. One of the students had written in a recent test, "… and so the equation is $+2$ and -2", which reminds us of Sergev and his clear understanding of equations, solutions of equations and solutions of problems modelled through equations.

The response to the demand of the teachers was to interrogate their talk when teaching equations. In most of my collaborative experiences of work with teachers, they did not normally feel that the mathematical richness of the classroom practices can be hampered by under specificity in talk, nor did they tend to feel that language was a content in mathematics teaching (Planas and Civil, 2009). Hence, the response was in a sense a surprise for Jana and Maia who were, as they said, expecting to engage in developmental work oriented to learn and practice mathematical tasks of explanation and modelling around the qualities and types of equations in the local curriculum. We finally agreed on exploring possibilities of improving content teaching of equations through improving teacher talk. For this, five 90-minute sessions were held. Even though the two teachers graduated in mathematics, there was initial time for revising mathematical knowledge on the equation concept and preceding the work driven by language-based tasks. There was a session organized around the task in Tab. 4, whose English version, with only some of the underlined examples of lexical elaborations produced during the session, does not intend to reproduce these sentences as if they were exactly equal in meaning to those discussed in Catalan. The sentences selected show choices in language that can inform mathematics teachers in the use of sentences for teaching equations in the secondary school. Although the original sentences from the lessons of Maia or Jana made good sense and could be said to work in teacher talk, they were not followed or preceded by complementary sentences adding content meaning, and were not placed into pedagogic general talk or application of routines. Even so, by presenting the sentences separated from the lesson context in which they were said, and whereas this was done intentionally in the design of the developmental task, the potential regarding newer meaning increased.

For each given sentence (left column, Tab. 4), the written practice was organized into individual writing, group discussion of the two individual proposals, and final shared writing on the worksheet. Jana and Maia decided on the mathematical meanings whose communication they wanted to prioritize in the re-elaboration of the original sentences (right column, Table 4). I pushed them to think of the individual writing as an opportunity for referring to the meanings that they missed most in test results and conversations with students, and that remained unfocused in the original sentences. We also reflected on the fact that, during their period of elaboration, mathematical procedures are conceptual in nature (Kieran, 2013). Some of the examples in Table 4 were published in Planas (2021b, pp. 282–283). The alternative sentences are not solutions in the sense of being totally adequate; they are just more appropriate in the sense of being closer to the idea of communicating mathematical meaning within languages of equations beyond the representation of operational routines. They are also more

Tab. 4. Examples of responses of the teachers in a workshop

What does the teacher say?	*What could the teacher say?*
We can solve a quadratic equation with a formula.	We can solve a quadratic equation with a formula. That is, we can obtain the numerical values for x that solve the equation.
We will modify the initial written equation.	We will modify the initial written equation. In other words, we will look for ways of writing the same equation for the final application of the formula.
Get a sequence.	Get a sequence, which is to say, get a sequence of equivalent equations, or equations with the same solutions.
Every equation, you change it a bit.	Every equation, you change it a bit. By changing it a bit, I mean adding, subtracting, multiplying or dividing both sides with the same numbers so that the solutions do not change.
You have to use the transposition rules.	You have to use the transposition rules. That is, the rules for the generation of equivalent equations.
You go mapping one written form to another up to the general formula on the board.	You go mapping one written form to another up to the general formula on the board. All the equations will be the same because the same numerical values solve them all.

appropriate in the sense of taking the opportunity to name relevant equation-related terms such as the names for the variable and the known and the unknown coefficient. In the final discussion, we talked about what we could possibly learn from the further elaborations of the original sentences. Jana said that even in the school lessons that are planned to practice the manipulation and resolution of equations, teacher talk can and must provide opportunities for students to step back and reflect upon what they are doing and why, in different lessons and teaching moments. Maia gave value to sentences with the names for quadratic equation, resolution and formula alongside descriptions such as "the formula for the resolution of a quadratic equation can be expressed in different ways using different letters for the variables." We discussed the questions of what is meant by the coefficients usually expressed with the letters a, b and c, and how this is told in relation to the concept of variable expressed with letters such as x, y and z. Regarding $ax^2 + bx + c = 0$, both teachers explained cases of students for whom a, b, c represented letters from which to generate numerical values. We concluded that the use of "letter" instead of coefficient and variable could be hindering conceptual learning. Two findings about knowledge gained by the teachers were:

- Direct naming of concepts through the use of specialized names in teaching offers opportunities to listen to important content meaning.
- Teaching talk that communicates content meaning in clear and precise forms increases opportunities of mathematical learning for understanding.

6. Conclusions and Ways Forward

In this paper, I have summarized sociocultural language-based research in mathematics education that has provided arguments for the importance of a notion of language as resource. This notion highlights the possibility that the access to and creation of opportunities for mathematics teaching and learning may be explained in part by the mathematical quality and clarity of the languages of teaching, rather than simply in terms of the capability of individual students or the skill of their teachers. The language as resource approach, as shaped by the arguments presented, encourages further investigation of the relationship between teaching languages and mathematical meaning communication in classroom encounters where participants use their languages for a diversity of purposes, activities and discourses. Few researchers have tried to relate the critical realization of some languages in school mathematics classrooms and the critical communication of mathematical meaning in teaching talk to access to and development of mathematics learning opportunities. Yet these relationships are of crucial interest and their understanding will possibly give rise to practical implications for all students, classrooms and mathematical contents.

The current state of sociocultural research on mathematics education and language suggests that strengthening and scaling up collaborations with teachers in schools and teacher educators in developmental settings may be part of the way forward. The last set of findings presented in the previous section, regarding developmental work with teachers to identify and enhance the potential of teaching talk to support mathematics learning for understanding, offers a promising path to follow in the direction of impacting positively on classroom practices and processes. Developmental work on the clarity and quality of content teaching talk over the course of classroom discourse practices, will enable mathematics teachers to focus on what they can do and say in teaching to increase their support for all students, regardless of their (everyday) languages, for mathematics content learning with understanding.

Acknowledgments

My thanks are for the students, families, teachers and schools who supported the studies, for my research group, GIPEAM, and for the Spanish and Catalan funding of Grants PID2019-104964GB-100 and EIN2019-103213, and SGR-2017-101.

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